

## Log - normal distribution of level curvatures in the localized regime: analytical verification

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1997 J. Phys. A: Math. Gen. 30 L339

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## LETTER TO THE EDITOR

**Log-normal distribution of level curvatures in the localized regime: analytical verification**Michail Titov<sup>†</sup>, Daniel Braun<sup>‡</sup> and Yan V Fyodorov<sup>†‡</sup><sup>†</sup> Petersburg Nuclear Physics Institute, Gatchina 188350, Russia<sup>‡</sup> Fachbereich Physik, Universität-GH Essen, D-45117 Essen, Germany

**Abstract.** We study numerically and analytically the moments of the dimensionless level curvature for one-dimensional disordered rings of the circumference  $L$  pierced by a magnetic flux  $\phi$ . The negative moments of the curvature distribution can be evaluated analytically in the extreme localization limit. The ensuing small curvature asymptotics of the corresponding distribution has a ‘log-normal’ behaviour. Numerically studied positive moments show differences from other log-normally distributed quantities.

Recently there has been a considerable interest in various statistical characteristics of spectra of disordered and chaotic quantum systems, see e.g. [1]. One particularly interesting issue is the so-called ‘level response statistics’ characterizing the sensitivity of individual energy levels  $E_n(\lambda)$  with respect to perturbation in an external parameter  $\lambda$ . The role of such a parameter can be played by, for example, an external electric or magnetic field, the strength and form of the potential, or any other appropriate parameter of different nature on which the system Hamiltonian is dependent. A convenient quantitative measure of level sensitivity is provided by a set of first and second derivatives  $\partial E_n(\lambda)/\partial\lambda$  and  $\partial^2 E_n(\lambda)/\partial\lambda^2$ , known as ‘level velocities’ and ‘level curvatures’, respectively.

The statistics of these quantities are mostly studied for the systems with completely ‘ergodic’ eigenfunctions covering randomly, but uniformly, all the available phase space and showing no specific internal structure. For systems of this type most of their statistical characteristics are known to be universal, i.e. independent on particular microscopic details, and adequately described by ensembles of large random matrices of particular global symmetry [1, 2]. The same universality class comprises also weakly-disordered (metallic) systems as long as effects of Anderson localization are negligible [1–3].

The form of the level curvature distribution typical for random matrices of various symmetry classes was guessed by Zakrzewski and Delande [4] on the basis of the numerical data. The analytical derivation of the Zakrzewski–Delande distribution is due to von Oppen [5] and Fyodorov and Sommers [6].

The effects of eigenfunction localization are expected to modify the level curvature statistics drastically. In contrast to the ‘random matrix’ regime it is obvious that the level response should be less universal depending on the nature of perturbation. Such a non-universality manifests itself already in the form of perturbative corrections to the curvature distribution due to localization effects [7].

Actually, the sensitivity of the energy spectrum to a change in boundary conditions was suggested long ago by Thouless as a measure of system conductance [8]. To quantify this

statement let us consider a one-dimensional sample closed to form a ring of circumference  $2L$  encircling the Aharonov–Bohm flux  $\phi$  (measured in units of the flux quanta  $\phi_0 = c\hbar/e$ ). The wavefunction in such geometry acquires the phase when going around the flux:

$$\Psi(x + L) = e^{i\phi}\Psi(x - L). \quad (1)$$

The curvature is given by second-order perturbation theory

$$K_n = \left. \frac{\partial^2 E_n(\phi)}{\partial \phi^2} \right|_{\phi=0} \propto \sum_{m \neq n} \frac{|\langle m | \hat{P}_x | n \rangle|^2}{E_m(0) - E_n(0)} \quad (2)$$

up to a constant shift. Here  $|m\rangle$  and  $E_m(0)$  are eigenvectors and eigenvalues of the Hamiltonian at zero flux and  $\hat{P}_x$  is the momentum operator. Using a similarity of this expression to the conductance given by the Kubo formula, Thouless argued that the ‘typical’ dimensionless level curvature  $K/\Delta$  (with  $K$  measured, e.g., by the widths of the curvature distribution) is proportional to the dimensionless conductance of the sample.

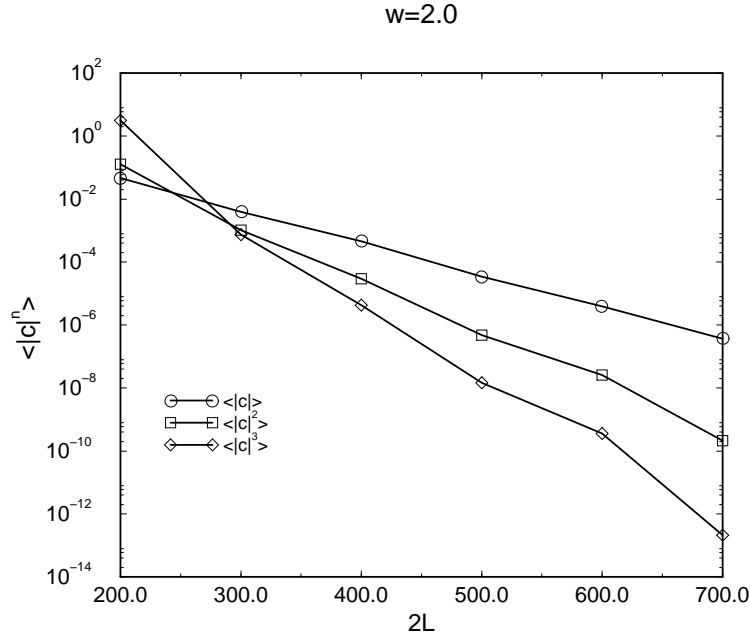
The original qualitative arguments of Thouless seemed to be controversial and also did not take into account strong correlations between energy levels of a disordered system. The curvature–conductance relation gave rise to a lot of discussion and was even claimed to be incorrect [9]. The problem has recently been reconsidered in much detail [10–12], the results favouring validity of the Thouless idea in the metallic regime. For a disordered system in a good metallic regime, the Thouless relation, including the universal proportionality coefficient, was derived explicitly [13] in the course of analytical verification of the Zakrzewski–Delande distribution. However, the perturbative localization corrections to both quantities are different [7].

As to the issue of the level curvatures in systems with strongly localized eigenstates, our present understanding is based mainly on the results of numerical simulations [12, 14–16]. It turns out that the curvature distribution is close to log–normal for strong enough localization and has quite a non-trivial form in the vicinity of the Anderson transition [15]. Qualitative origin of the log–normal distribution can be inferred from equation (2) as suggested in [15]. Indeed, it is natural to assume that the absolute value of the curvature  $|K|$  is proportional to the product of amplitudes of a typical wavefunction at opposite edges  $\pm L$  of the sample:  $|K| \propto |\Psi(-L)\Psi(L)| \sim \exp(-\text{constant} \times L/\xi)$ . The log–normal distribution follows if one assumes that the inverse localization length  $\xi^{-1}$  shows Gaussian eigenstate-to-eigenstate fluctuations.

However attractive and transparent this simple argumentation, a close inspection shows that it should be taken with caution. To this end let us recall that the distribution of the quantity  $v = |\Psi(-L)\Psi(L)|$  for (quasi) one-dimensional disordered samples can be calculated analytically [17]. It indeed has a log–normal form for  $L \gg \xi$ , but all the positive moments  $\langle v^n \rangle$  are dominated by rare events in such a way that  $\langle v^n \rangle \propto \exp(-cL/\xi)$ , where the constant  $c$  is *independent* of the index  $n$ .

To check if this property is shared by the level curvatures, we performed numerical simulations of the tight-binding Anderson model in one dimension. The curvatures were calculated exactly from both eigenvalues and eigenvectors (see figure 1) and between 100 and 500 disorder realizations were used. We plot the numerical results for the logarithm of first, second and third moments of the curvature distribution against  $L$ , see figure 1. The moments indeed decay roughly exponentially with system size, but the typical decay length decreases for higher moments in contrast to the behaviour typical for the quantity  $|\Psi(-L)\Psi(L)|$ .

The second fact to be mentioned is that earlier numerical investigations discovered a peculiar feature of the curvature distribution: the log–normal law is quite a good fit



**Figure 1.** First, second and third moments of the curvature distribution as functions of the system size as obtained from numerical simulations of the tight-binding Anderson model in one dimension. The disorder parameter  $w$  (width of the box distribution of the diagonal matrix elements) is  $w = 2$ . Typical error bars are comparable to the remaining fluctuations on top of the exponential decay.

separately for the domain of curvatures larger and smaller than the most probable value  $\langle \ln |K| \rangle \sim L/\xi$ , but the parameters of the fit are slightly different for the two domains. At the same time, the statistics of the correlation function  $v = |\Psi(-L)\Psi(L)|$  are truly log-normal everywhere [17].

These observations provide us with a motivation to consider the problem of the level curvature distribution in a strongly localized regime on a more sound basis without invoking any additional assumptions.

In the present paper we treat analytically the problem of the distribution of the absolute value of the level curvatures:

$$\mathcal{P}(K) = \Delta \left\langle \sum_n \delta(E - E_n(0)) \delta(K - |K_n|) \right\rangle \quad (3)$$

for a one-channel ring characterized by the Schrödinger equation

$$\left( -\frac{d^2}{dx^2} + U(x) - E \right) \Psi(x) = 0 \quad (4)$$

with the boundary condition equation (1). Here  $U(x)$  is a white noise potential  $\langle U(x)U(x') \rangle = D\delta(x - x')$  which is considered to be weak,  $l = 4k^2/D \gg k^{-1}$ , with  $k$  being the Fermi momentum related to the energy  $E$  as  $k^2 = E$  and  $l$  standing for the mean free path. For the one-channel ring the mean free path is of the same order as the localization length. As we shall see, the negative moments of the curvature distribution equation (3) can be found explicitly for rings with  $L \gg l$ , i.e. in a strong localization regime.

This allows us to verify analytically the log-normal nature of the ensuing distribution at small curvatures.

To address the problem of level curvatures in the most direct way we follow the method of Dorokhov [18] and Kolokolov [19] who calculated the absolute value of the persistent current  $j_n = |\partial E_n / \partial \phi|$ . To this end, let us associate with any point  $x$ ,  $-L \leq x \leq L$ , within the sample a vector  $\mathbf{V}(x) = \begin{pmatrix} v_+(x) \\ v_-(x) \end{pmatrix}$  with components  $v_{\pm}(x) = \pm(d\Psi/dx \pm ik\Psi) \exp(\mp ikx)$  and consider a  $2 \times 2$  transfer matrix  $\mathcal{T}$  relating the value  $\mathbf{V}(x)$  to the 'initial' value  $\mathbf{V}(-L)$  in the following way:  $\mathbf{V}(x) = \hat{\mathcal{T}}(x, -L)\mathbf{V}(-L)$ . Due to current conservation the transfer matrix can be parametrized as

$$\hat{\mathcal{T}}(x, -L) = \begin{pmatrix} \cosh \Gamma e^{i\alpha} & \sinh \Gamma e^{i\beta} \\ \sinh \Gamma e^{-i\beta} & \cosh \Gamma e^{-i\alpha} \end{pmatrix}. \quad (5)$$

Here  $\alpha(x, E)$ ,  $\beta(x, E)$  and  $\Gamma(x, E)$  are real functions to be determined. The periodic boundary condition equation (1) can be written in terms of the transfer matrix  $\hat{\mathcal{T}} = \hat{\mathcal{T}}(L, -L)$  as  $\det(\hat{\mathcal{T}} e^{2(ikL\hat{\sigma}_z)} - e^{i\phi}) = 0$ , where  $\hat{\sigma}_z$  stands for the Pauli matrix. It is convenient to rewrite this condition in terms of the  $\hat{\mathcal{T}}$ -matrix elements as

$$f(E_m, L) = \cos \phi \quad f(E, x) = \cosh \Gamma \cos(\alpha + 2kx). \quad (6)$$

This equation determines the set of energy levels  $E_m(\phi)$ . The following identities can be immediately inferred from this fact:

$$\sum_n \delta(E - E_n(\phi)) = \delta(f(E, L) - \cos \phi) |df/dE| \quad (7)$$

$$\left| \frac{\partial^2 E_n}{\partial \phi^2}(\phi = 0) \right| = \left| \frac{\partial f}{\partial E}(E = E_n) \right|^{-1}. \quad (8)$$

As a result one can write down the moments of the distribution equation (3) in the following form:

$$M_n = \int_0^\infty K^n \mathcal{P}(K) = \frac{1}{\Delta^{n-1}} \left\langle \delta(f - 1) \left| \frac{\partial f}{\partial E} \right|^{-n+1} \right\rangle. \quad (9)$$

When performing the disorder averaging it is convenient to get rid of the  $\delta$ -functions in (9) by averaging the corresponding expression over the ensemble of samples with slightly fluctuating sample lengths  $L \pm \delta L$ ;  $k^{-1} \delta L \ll \min(l, L)$  [19]. When doing this we take into account the fact that the functions  $\Gamma(x, E)$  and  $\alpha(x, E)$  change very slowly on the scale  $\delta x \sim k^{-1}$ , i.e.  $(d\Gamma/dx, d\alpha/dx) \ll k$ . As a consequence there is typically only one solution of equation (6) in the interval  $\delta x = \pi/k$ . We also found that the quantities  $d\alpha/dE$  and  $Lk^{-1}$  are of the lower order  $O(1/(Lk))$  when compared with  $d\Gamma/dE$  and can be safely neglected. This gives the possibility of rewriting the expression (9) in the form

$$M_n \approx \frac{1}{2\pi \Delta^{n-1}} \left\langle \frac{1}{|\sinh \Gamma|} \left| \frac{d\Gamma}{dE} \tanh \Gamma \right|^{-n+1} \right\rangle. \quad (10)$$

The disorder averaging in expressions of such a type can be performed by employing the functional integral method suggested by Kolokolov [19]. For accomplishing such a calculation it is important to express the quantity to be averaged in terms of elements of the matrices  $\mathcal{T} \equiv \mathcal{T}(L, -L)$  and  $d\mathcal{T}/dE$  without involving complex conjugation. To find such a representation let us consider the auxiliary quantity

$$A = (0, 1) \mathcal{T} s^- s^+ \mathcal{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (11)$$

where  $s^\pm$  are familiar lowering and raising operators for the spin  $\frac{1}{2}$ . Taking the parametrization equation (5) into account we find that  $A = \cosh^2 \Gamma e^{-2i\alpha}$  and correspondingly

$$\frac{dA/dE}{2A} = \frac{d\Gamma}{dE} \tanh \Gamma - i \frac{d\alpha}{dE} = \frac{d\Gamma}{dE} \tanh \Gamma \left( 1 + O\left(\frac{1}{Lk}\right) \right). \quad (12)$$

The right-hand side of this expression is just the important part of the combination appearing in expression (10) under the sign of averaging.

On the other hand, the same quantity  $(dA/dE)/2A$  can be written in terms of the matrix  $dT/dE$  when differentiating equation (11) with respect to  $E$ .

The matrix  $dT/dE$  itself can be determined from the following exact relation between the solution  $\Psi(x)$  of the initial Schrödinger equation and its derivative with respect to the energy  $\Phi(x) = d\Psi(x)/dE$ ,

$$\Phi(x) = \Psi(x) \left[ c_1 - \int_{-L}^x \frac{dy}{\Psi^2(y)} \left( c_0 + \int_{-L}^y dy_1 \Psi^2(y_1) \right) \right] \quad (13)$$

where  $c_0 = (\Phi d\Psi/dx - \Psi d\Phi/dx)|_{x=-L}$  and  $c_1 = \Phi(-L)/\Psi(-L)$ .

We can write the relation

$$\frac{dT}{dE} \mathbf{V}(-L) = \mathbf{F}(L) - \mathcal{T} \mathbf{F}(-L) \quad (14)$$

where  $\mathbf{F}(x) = d\mathbf{V}(x)/dE$ . The components  $f_\pm$  of the vector  $\mathbf{F}$  are represented as integral functionals of the fields  $v_\pm(x)$ , the latter fields having boundary values  $v_\pm(-L)$  at the point  $x = -L$ .

Combining all these facts together, one can find the appropriate representation for the quantity  $(dA/dE)/2A$ . Skipping the cumbersome intermediate steps in favour of presenting the final expression, we find finally

$$\frac{dA/dE}{2A} \simeq \int_{-L}^L \frac{dy}{v_+(y)v_-(y)} \int_{-L}^y v_+(y_1)v_-(y_1) dy_1 \quad (15)$$

where the components  $v_\pm(x)$  of the field  $\mathbf{V}(x)$  are given by  $\mathbf{V}(x) = \mathcal{T}(x, -L) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . In expression (15) we only kept the leading terms with respect to  $1/Lk$  and also omitted all terms containing the fast oscillating factors  $e^{\pm 2ikL}$ .

The following comment is appropriate here. Generally speaking, the functions  $v_\pm(x)$  are random and can take any complex values. However, they are not independent of each other and can be chosen to satisfy the constraint  $v_+(x) = v_-^*(x)$ . When using the Kolokolov method of averaging the random fields,  $v_+(x)$  and  $v_-(x)$  should be analytically continued on the surface determined by this constraint to provide the convergence of the corresponding path integral. Apart from this fact, in our case one can use the same constraint to argue that  $d\alpha/dE$  is small in comparison with  $\tanh \Gamma d\Gamma/dE$  by the factor  $1/Lk$  which we intensively used to simplify the expressions given above, for example equation (12).

Equation (10) combined with (12)–(15) provides a representation which is used as a starting point for employing the Kolokolov approach. Indeed, we expressed the moments  $M_n$  in terms of the elements of the  $\mathcal{T}$ -matrix without complex conjugation: the components of  $v_\pm(x)$  were related to the  $\mathcal{T}$ -matrix above and  $|\sinh \Gamma|$  can be represented in terms of the  $\mathcal{T}$ -matrix in the same way as in [19].

After the set of manipulations identical to those used in [19] we represent the moments in the form of the following path integral:

$$M_n = \frac{2e^{-c/4}}{\pi(l^2\Delta)^{n-1}} \int_{-\infty}^{\infty} d\sigma d\sigma' \int_{\xi(-L)=\sigma'; \xi(L)=\sigma} \mathcal{D}\xi e^{-\xi(-c)/2} \exp\left\{-\frac{1}{2} \int_{-c}^c dx (\dot{\xi}^2 + e^{-\xi})\right\} \times \left[ \int_{-c}^c e^{-\xi(s)} ds \int_{-c}^s e^{\xi(p)} dp \right]^{-n+1} \quad (16)$$

where  $c \equiv 2L/l$  is the relation between the ring circumference  $2L$  and the mean free path  $2l$ .

In general, only the first moment ( $n = 1$ ) related to the average persistent current can be explicitly evaluated for arbitrary  $c$  [18, 19].

However, in the limit  $2L \gg l$  we can extract the leading contribution to the path integral (16) for a large, but limited, number of negative moments satisfying  $|n| \lesssim L/l$ . To see this let us use the notation  $m = -n$  so that the twofold integral in equation (16) is raised to the positive power  $m + 1$ . This gives a possibility of treating the path integral in equation (16) as the sum of  $2(m + 1)$ -fold integrals from the corresponding matrix elements. To illustrate the structure of the terms we write down explicitly the simplest one:

$$M_0 = \frac{2e^{-c/4}}{\pi(l^2\Delta)^{n-1}} \int_{-c}^c ds \int_{-c}^s dp \left\langle 1 \left| e^{-(c-s)\hat{\mathcal{H}}_t} e^{-(s-p)\hat{\mathcal{H}}_t} \frac{1}{t^2} e^{-(p+c)\hat{\mathcal{H}}_t} \right| t \right\rangle \quad (17)$$

where we introduced the new variable  $t = 2e^{-\xi/2}$  in terms of which the Hamiltonian operator  $\hat{\mathcal{H}}$  and the scalar product  $\langle f|g \rangle$  read  $\hat{\mathcal{H}} = -\frac{1}{8}(t^2\partial_t^2 + t\partial_t - t^2)$  and  $\langle f|g \rangle = \int_0^\infty (dt/t) fg$ , correspondingly.

Higher moments will contain similar integrals over variables  $-c < s_1, p_1, s_2, p_2, \dots, s_m, p_m < c$ . Each integral of this type can be evaluated analytically if we expand powers of  $t$  in terms of the eigenfunctions of the operator  $\mathcal{H}$  which are the modified Bessel functions:  $\mathcal{H}K_p(t) = -1/8p^2K_p(t)$ . The functions with imaginary indices  $p = iv$  form a complete orthogonal set suitable for expansion. In this way we find that the leading contribution to the path integral (16) in the limit  $L \gg l$  corresponds to the configuration  $p_1 \approx p_2 \approx \dots \approx p_m = -c; s_1 \approx \dots \approx s_m = c$  when we can effectively write that

$$\left[ \int_{-c}^c e^{-\xi(s)} ds \int_{-c}^s e^{\xi(p)} dp \right]^{m+1} \approx [(m+1)!]^2 e^{-(m+1)\xi(c)} e^{(m+1)\xi(-c)} \quad (18)$$

for  $0 \leq m \ll c/2$ . For larger  $m$  the contribution from omitted terms becomes comparable with that given by equation (18) due to large combinatoric factors.

The moments  $M_{-m}$  in this approximation are equal to

$$M_{-m} = \frac{1}{m\sqrt{\pi}} e^{-c/4} [(m+1)!]^2 \Gamma\left(2m + \frac{3}{2}\right) e^{c(m+1/2)^2}. \quad (19)$$

When restoring the asymptotic behaviour of the curvature distribution from these moments the factorial coefficients can be omitted in view of the condition  $m \ll c/2$ . This results in the log-normal curvature distribution

$$\mathcal{P}(\ln K) \simeq (4\pi c)^{-1/2} e^{-(1/4c)[\ln K + c]^2} \quad (20)$$

which is valid in the strongly localized limit  $c \gg 1$  inside the parametrically large domain of small curvatures  $-c^2 \ll \ln K \ll -c$ , where  $c = 2L/l$ .

In conclusion, we have demonstrated explicitly that the statistics of level curvatures in one-dimensional disordered systems is log-normal inside a parametrically large domain of curvatures smaller than the typical value. At the same time, interesting issues of explaining

an unexpected difference in behaviour of the positive moments of the curvature and that of the eigenfunction correlator, see figure 1, as well as the mentioned asymmetry of the curvature distribution remain open.

The authors are grateful to H-J Sommers for his valuable critical comments, I Kolokolov for important clarifying communications in the process of research and V Kravtsov for useful correspondence and interest in the work. The financial support by the Russian Foundation for Basic Researches (grant No 96-02-18037a) (MT), program 'Quantum Chaos' (grant No INTAS-94-2058) (MT and YVF), and by SFB 237 'Unordnung and Grosse Fluktuationen' is acknowledged with thanks.

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